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Asymptotic solutions for the asymmetric channel flow with porous retractable walls and a transverse magnetic field

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Abstract

The self-similarity solutions to the Navier-Stokes equations are constructed for an incompressible laminar flow through a uniformly porous channel with retractable walls under a transverse magnetic field. The flow is driven by the expanding or contracting walls with different permeability. The velocities of the asymmetric flow at the upper and lower walls are different in not only the magnitude but also the direction. The asymptotic solutions are well constructed with the method of boundary layer correction in two cases with large Reynolds numbers, i.e., both walls of the channel are with suction, and one of the walls is with injection while the other one is with suction. For small Reynolds number cases, the double perturbation method is used to construct the asymptotic solution. All the asymptotic results are finally verified by numerical results.

Keywords: laminar flow, asymmetric flow, asymptotic solution, porous and retractable channel, magnetic field

1. Introduction

The studies of laminar flow through porous channels with retractable walls have received extensive attention in the fields of fluid and mathematics due to their close connections with massive biological and engineering problems, e.g., blood flow in vessels, nutrition liquid transport in biological organisms, mass transfer among blood, air, and tissue, uniformly distributed irrigation, and natural transpiration. The electrically conducting viscous fluid in a channel with permeable walls can be used to simulate the biological problems. Taking blood flow in vessels as an example, when the blood is considered as an electrically conducting fluid, Higashi et al.[1] showed that the magnetic field has a significant effect on the vascular system based on the experimental investigation.

Berman [2] analyzed the two-dimensional steady laminar flow of a viscous incompressible fluid through a porous channel with uniform injection or suction by full using the symmetry to simplify the model for the first time, and reduced the Navier-Stokes equations to a fourth-order nonlinear ordinary differential equation with a parameter R (see Section 2 for its definition) and four boundary conditions. Numerous studies about the laminar flow in a channel with permeable walls followed. Yuan [3] derived an asymptotic solution for the large injection case. Terrill [4] improved the solution by considering the inner layer. Sellars [5] and Terrill [6] investigated the large suction cases, and derived an asymptotic solution. Afterwards,

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in order to precisely simulate the blood flow, Uchida and Aoki [7] investigated the unsteady laminar flow driven by a single contraction or expansion of the channel walls. Goto and Uchida [8] studied the laminar flow in a semi-infinite porous pipe, the radius of which varied with time. Majdalani et al.[9] studied the laminar flow between slowly expanding or contracting walls, and obtained the similarity solution to describe the transport of biological fluids. Majdalani and Zhou [10] investigated the large injection and suction cases in a channel with retractable walls, and obtained the asymptotic solutions. For more details, one may refer to Asghar et al.[11], Xu et al.[12], and Dauenhauer and Majdalani [13].

Suryaprakasrao [20] investigated the laminar flow of an electrically-conductive viscous fluid in a porous channel under a transverse magnetic field for the first time, and obtained the asymptotic solution for the case with a small suction Reynolds number and small magnetic field number. Terrill and Shrestha [21, 22, 23] and made further extension of Suryaprakasrao's work, and obtained the asymptotic solutions for large suction and injection Reynolds numbers and all values of the Hartamnn number. Based on these works, investigators began to take account of the wall motion [24].

The asymmetric laminar flow caused by different wall permeability can be traced back to Proudman [14] , who proposed the asymmetric flow for the first time. Terrill and Shrestha [15, 16, 17] and Shrestha and Terrill [17] extended Proudman'work, and obtained a series of asymptotic solutions with the method of matched asymptotic expansions for large injection, large suction, and mixed cases. Cox [18] and King and Cox [19] considered the problem of steady and unsteady flow in a channel with only one porous wall. Zhang et al.[26] studied the asymmetric flow analytically and numerically. However, these asymmetric work did not consider the cases with wall motion and a transverse magnetic filed in a channel.

In this paper, we will investigate the general asymmetric flow of an incompressible viscous fluid through a porous and retractable channel with a transverse magnetic field, and present the asymptotic solutions. The paper is arranged as follows. In Section 2, the formulation of the problem is presented by reducing the Navier-Stokes equations into a nonlinear ordinary differential equation via a similarity transformation. In Section 3, the effects of the magnetic field on the solution is examined, and the asymptotic solutions for different orders of the Reynolds and Hartman numbers are obtained. In Section 4, the asymptotic solutions for large Reynolds numbers in the case that one wall of the channel is with injection while the other wall is with suction are constructed. In Section 5, an asymptotic solution for the case of slowly contracting and weak permeability is presented. In Section 6, all the obtained solutions are verified by the numerical solutions. The summarization is given in Section 7 finally.

2. Mathematical formulation

The equations of the continuity and momentum for the unsteady laminar flow of an incompressible viscous and electrically conducting fluid through a porous and retractable channel with a transverse magnetic field are

$$\nabla \cdot \boldsymbol{V} = 0, \tag{2.1}$$

and

$$\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V} \cdot \nabla) \boldsymbol{V} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \boldsymbol{V} + \frac{1}{\rho} \boldsymbol{J} \times \boldsymbol{B}, \tag{2.2}$$

where $\boldsymbol{J}$ and $\boldsymbol{B}$ are given by the Maxwell equations

$$\nabla \times \boldsymbol{H} = 4\pi \boldsymbol{J}, \tag{2.3}$$

$$\nabla \times \boldsymbol{E} = 0, \tag{2.4}$$

$$\nabla \cdot \boldsymbol{B} = 0, \tag{2.5}$$

and Ohm's law

$$\boldsymbol{J} = \sigma(\boldsymbol{E} + \boldsymbol{V} \times \boldsymbol{B}). \tag{2.6}$$

In the above equations, $\boldsymbol{B} = \mu_m \boldsymbol{H}$. The symbol $\boldsymbol{J}$ represents the current density, $\boldsymbol{E}$ the electric field, $\boldsymbol{H}$ the magnetic field, ν the viscosity of the fluid, σ the electrical conductivity and μ_m the magnetic permeability. An elongated rectangular channel exhibiting a sufficiently large aspect ratio of width L to height $h(t)$ and with one closed end is here considered. Despite the channel's finite body length, it is reasonable to assume that the channel is semi-infinite length in order to neglect the influence of the opening at the end[7]. Permeability of both walls which expand or contract uniformly at a time-dependent rate $\dot{h}(t)$ is different. Assume that the fluid velocity at the lower wall is $-v_1$ and the upper wall is $-v_2$. Furthermore, a constant magnetic field of strength H_0 is applied perpendicular to the walls. It is assumed that there is no external electric field and the effect of magnetic and electric field produced by the motion of the electrically conducting fluid can be negligible[20, 21, 22]. With these assumptions the magnetic term $\boldsymbol{J} \times \boldsymbol{B}$ of the body force in (2.2) reduces to

$$\boldsymbol{J} \times \boldsymbol{B} = -\sigma {B_0}^2 \boldsymbol{V}, \tag{2.7}$$

where $B_0 = \mu_m H_0$. Let $\widetilde{x}$ and $\widetilde{y}$ be the co-ordinates measured along and perpendicular to the flow direction respectively, u and v be the velocity components ($i.e.\, \boldsymbol{V} = (u, v)$) in the $\widetilde{x}$ and $\widetilde{y}$ directions respectively.

The boundary conditions satisfied by the flow are

$$\begin{aligned} u(\widetilde{x}, -h) &= 0, & v(\widetilde{x}, -h) &= -v_1 = -A_1 \dot{h}, \\ u(\widetilde{x}, h) &= 0, & v(\widetilde{x}, h) &= -v_2 = -A_2 \dot{h}, \end{aligned} \tag{2.8}$$

where A_1 and A_2 are constant measures of the lower and upper wall permeabilities, respectively. We shall introduce a stream function[10]

$$\phi = \frac{\nu \widetilde{x}}{h} F(y, t), \tag{2.9}$$

where $y = \widetilde{y}/h$ is the dimensionless height. Then the velocity components are given by

$$u = \frac{\partial \phi}{\partial \widetilde{y}} = \frac{\nu \widetilde{x}}{h^2} F_y, \qquad v = -\frac{\partial \phi}{\partial \widetilde{x}} = -\frac{\nu}{h} F, \tag{2.10}$$

so that the continuity equation (2.1) is naturally satisfied. Assuming $|v_2| \geq |v_1|$, without loss of generality, and substituting (2.10) into (2.1)-(2.2) give rise to

$$f''' + \alpha(y f'' + 2f') + R(f f'' - f'^2) - M^2 f' = K(R), \tag{2.11}$$

with the boundary conditions

$$f(-1) = 1 - \alpha_2, \quad f'(-1) = 0, \quad f(1) = 1, \quad f'(1) = 0, \tag{2.12}$$

where α is the wall expansion ratio defined by $\alpha = \frac{h\dot{h}}{\nu}$ (positive for expansion and negative for contraction), $R = \frac{v_2 h}{\nu}$ is the Reynolds number (positive for injection and negative for suction), $M = \mu_m H_0 h (\frac{\sigma}{\rho\nu})^{\frac{1}{2}}$ is the Hartman number, $\alpha_2 = 1 - \frac{v_1}{v_2}$ is an asymmetric parameter and K is an integration constant. Similar derivations for transforming the Navier-Stokes equation into (2.11) can be referred to Majdalani and Zhou[9]. The fluid flow through the channel is symmetric about the centre line of the channel for $\alpha_2 = 2$. In this paper, we will mainly focus on the asymmetric flow.

3. Asymptotic solutions for large suction case

In this section, we will consider the case that both walls of the channel are with large suction and the suction velocities on the upper and lower wall are different. An asymptotic solution will be constructed for $\alpha = O(1)$ and $M^2 = O(1)$ as $R \to -\infty$ and for $\alpha = O(1)$ and $M^2 = O(R)$ as $R \to -\infty$, respectively.

3.1. Asymptotic solution for large suction Reynolds number R

For the large suction case, assuming $v_1 > 0 > v_2$, by treating $\epsilon = -\frac{1}{R} > 0$ as a perturbation parameter, (2.11) can be written as

$$\epsilon f''' + \epsilon\alpha(yf'' + 2f') - (ff'' - f'^2) - \epsilon M^2 f' = k, \tag{3.1}$$

where $\varepsilon K(R) = k$, which is the equation to be solved for the suction case subject to the boundary conditions (2.12). If the channel wall is with large suction, there exists a boundary layer near the upper wall for $M = 0$ and $\alpha_2 = 2$[10]. Motivated from their thinking, there may be a boundary layer near both walls and the correction near both walls may be needed. Therefore, the method of boundary layer correction[25] is used to construct the asymptotic solution. Hence, the solution can be expanded as following form

$$f = f_0(y) + \epsilon(f_1(y) + g_1(\tau) + h_1(\eta)) + \epsilon^2(f_2(y) + g_2(\tau) + h_2(\eta)) + \cdots, \tag{3.2}$$

and the integrating constant k can be written as

$$k = k_0 + \epsilon k_1 + \epsilon^2 k_2 + \cdots, \tag{3.3}$$

where $\tau = \frac{1-y}{\epsilon}$ and $\eta = \frac{1+y}{\epsilon}$ are the stretching transformations near the wall dimensionless height $y = 1$ and $y = -1$, respectively, and $g_i(\tau)$ and $h_i(\eta)$, $i = 1, 2, 3\cdots$, are boundary layer functions ($g_i(\tau)$ and $h_i(\eta)$ rapidly decay when y is away from $y = 1$ and $y = -1$, respectively).

Substituting (3.2) and (3.3) into (3.1) and equating the equal powers of ϵ yield

$$\epsilon^0 : f_0'^2 - f_0 f_0'' = k_0 \tag{3.4}$$

$$\epsilon^1 : 7f_0 f_1'' - 2f_0' f_1' + f_0'' f_1 = \alpha(yf_0'' + 2f_0') - M^2 f_0' + f_0''' - k_1 \tag{3.5}$$

$$\epsilon^2 : f_0 f_2'' - 2f_0' f_2' + f_0'' f_2 = \alpha(yf_1'' + 2f_1') - M^2 f_1' + f_1''' - f_1 f_1'' + f_1'^2 - k_2 \tag{3.6}$$

$\cdots\cdots$

$$\epsilon^{-1} : \dddot{g}_1 + f_0(1)\ddot{g}_1 = 0 \tag{3.7}$$

$$\epsilon^0 : \dddot{g}_2 + f_0(1)\ddot{g}_2 = (\alpha - f_1(1))\ddot{g}_1 + f_0'(1)\tau\ddot{g}_1 - g_1\ddot{g}_1 - 2f_0'(1)\dot{g}_1 + \ddot{g}_1^2 = 0 \tag{3.8}$$

$\cdots\cdots$

$$\epsilon^{-1} : \dddot{h}_1 - f_0(-1)\ddot{h}_1 = 0 \tag{3.9}$$

$$\epsilon^0 : \dddot{h}_2 - f_0(-1)\ddot{h}_2 = (f_1(-1) + \alpha)\ddot{h}_1 + f_0'(-1)\eta\ddot{h}_1 + h_1\ddot{h}_1 - 2f_0'(-1)\dot{h}_1 - \dot{h}_1^2 = 0 \tag{3.10}$$

$\cdots\cdots$

where f', $\dot{g}$ and $\dot{h}$ denote the derivatives with respect to y, τ and η, respectively, and we have used $f_i(y) = f_i(1-\epsilon\tau) = f_i(1) - \epsilon\tau f_i'(1) + \frac{1}{2}\epsilon^2\tau^2 f_i''(1) + \cdots$ and $f_i(y) = f_i(\epsilon\eta - 1) = f_i(-1) + \epsilon\eta f_i'(-1) + \frac{1}{2}\epsilon^2\eta^2 f_i''(-1) + \cdots$, $i = 0, 1, 2\cdots$, and $g_j(\tau)h_j(\eta)$ is exponentially small (considering them approximately as zero in the rest of the section, $j = 1, 2\cdots$). By substituting (3.2) into (2.12), the boundary conditions to be satisfied by

$f_i(y)$, $g_i(\tau)$ and $h_i(\eta)$ at $y = 1$ or $\tau = 0$ and $y = -1$ or $\eta = 0$ are

$$f_0|_{y=1} = 1,\ f_0|_{y=-1} = 1 - \alpha_2 = a, \tag{3.11}$$

$$f'_{i-1}|_{y=1} - \dot{g}_i|_{\tau=0} = 0,\ i = 1, 2, \cdots, \tag{3.12}$$

$$f'_{i-1}|_{y=-1} + \dot{h}_i|_{\eta=0} = 0,\ i = 1, 2, \cdots, \tag{3.13}$$

$$f_i|_{y=1} + g_i|_{\tau=0} = 0,\ f_i|_{y=-1} + h_i|_{\eta=0} = 0,\ i = 1, 2, \cdots, \tag{3.14}$$

where $a = \frac{v_1}{v_2}$ (i.e.$-1 < a < 0$). One solution of (3.4) with the boundary condition (3.11) is

$$f_0 = \frac{1-a}{2}y + \frac{1+a}{2} \tag{3.15}$$

and $k_0 = \frac{(a-1)^2}{4}$. Thus, the boundary condition (3.12) becomes

$$\dot{g}_1|_{\tau=0} = f'_0|_{y=1} = \frac{1-a}{2}. \tag{3.16}$$

The boundary layer solution of (3.7) satisfying the condition (3.16) is

$$g_1 = \frac{a-1}{2}e^{-\tau}. \tag{3.17}$$

Also, the boundary layer solution of (3.9) satisfying the condition (3.13) is

$$h_1 = \frac{a-1}{2a}e^{a\eta}. \tag{3.18}$$

Substituting (3.15) into (3.5) gives rise to

$$\frac{1}{2}((a-1)y - a - 1)f''_1 + (1-a)f'_1 = \frac{1}{2}(2\alpha - M^2)(a-1) - k_1. \tag{3.19}$$

The corresponding boundary conditions from (3.14) are

$$f_1|_{y=1} = -g_1|_{\tau=0} = \frac{1-a}{2},\ f_1|_{y=-1} = -h_1|_{\eta=0} = \frac{1-a}{2a}. \tag{3.20}$$

Hence, the solution of f_1 subject to (3.20) is

$$\begin{aligned} f_1 =& \frac{-1}{16a(a^2+a+1)}\{[(a-1)^2(a^2+2M^2a-4\alpha a-2a+1) - 4a(a-1)k_1]y^3 - 3[(a^2-1)(a^2 \\ &+ 2M^2a - 4\alpha a - 2a + 1) - 4a(a+1)k_1]y^2 + [(a-1)^2(3a^2 - 2M^2a + 4\alpha a + 6a + 3) \\ &4a(a-1)k_1]y + [(a^2-1)(7a^2 + 6M^2a - 12\alpha a - 2a + 7) - 12a(a+1)k_1]\}, \end{aligned} \tag{3.21}$$

where the parameter k_1 is still unknown and to be determined next.
Substituting (3.15) and (3.21) into (3.6), one can obtain the expression of f_2. We find that one term of f_2 is

$$\frac{(M^2-4\alpha)[(a^2+2M^2a-4\alpha a-2a+1)(a-1) - 4ak_1]}{8(a-1)^3(a^2+a+1)a}((1-a)y+a+1)^3\log((1-a)y+a+1), \tag{3.22}$$

which is a secular term. Thus, the term must be zero and then $k_1 = \frac{(a-1)}{4a}(a^2 + 2M^2a - 4\alpha a - 2a + 1)$.

The expression of f_1 becomes

$$f_1 = -\frac{(a-1)^2}{4a}y + \frac{1-a^2}{4a}. \tag{3.23}$$

Then, according to (3.12), the boundary condition for g_2 becomes

$$\dot{g}_2|_{\tau=0} = f_1'|_{y=1} = \frac{-(a-1)^2}{4a}. \tag{3.24}$$

Substituting (3.17) into equation (3.8), we obtain

$$g_2 = \frac{1-a}{8a}[(a^2-a)\tau^2 + 2(3a^2 - 2\alpha a - 3a)\tau + 6a^2 - 4\alpha a - 8a + 2]e^{-\tau}. \tag{3.25}$$

Similarly, the solution of h_2 becomes

$$h_2 = \frac{a-1}{8a^3}[-(a^3-a^2)\eta^2 + 2(2\alpha a + 3a - 3)a\eta + 2(a^2 - 2\alpha a - 4a + 3)]e^{a\eta}. \tag{3.26}$$

Hence, the boundary conditions for f_2 from (3.14) are

$$f_2|_{y=1} = -g_2|_{\tau=0} = \frac{(a-1)}{4a}(3a^2 - 2\alpha a - 4a + 1), \tag{3.27}$$

$$f_2|_{y=-1} = -h_2|_{\eta=0} = \frac{(1-a)}{4a^3}(a^2 - 2\alpha a - 4a). \tag{3.28}$$

Substituting (3.15) and (3.23) into (3.6) yields

$$\frac{1}{2}((a-1)y - a - 1)f_2'' + (1-a)f_2' + \frac{(a-1)^2}{16a^2}(a^2 + (4M^2 - 8\alpha - 2)a + 1) = k_2. \tag{3.29}$$

One obtains

$$f_2 = \frac{a-1}{8a^3}[(3a^4 - (4+2\alpha)a^3 + 2a^2 - (4+2\alpha)a + 3)y + (a^2-1)(3a^2 - (4+2\alpha)a + 3)]? \tag{3.30}$$

and $k_2 = -\frac{(a-1)^2}{16a^3}(6a^4 - (9+4\alpha)a^3 - (4M^2 - 8\alpha - 6)a^2 - (9+4\alpha)a + 6)$ which is determined by eliminating the Secular term of f_3. Finally, the asymptotic solution for the large suction case of $y \in [-1, 1]$ is

$$\begin{aligned} f(y) =& \frac{1-a}{2}y + \frac{1+a}{2} + \epsilon[-\frac{(a-1)^2}{4a}y + \frac{1-a^2}{4a} + \frac{a-1}{2}e^{-\tau} + \frac{a-1}{2a}e^{a\eta}] + \epsilon^2\{\frac{a-1}{8a^3}[(3a^4 - (4 \\ &+ 2\alpha)\cdot a^3 + 2a^2 - (4+2\alpha)a + 3)y + (a^2-1)(3a^2 - (4+2\alpha)a + 3)] + \frac{1-a}{8a}[(a^2-a)\tau^2 \\ &+ 2(3a^2 - 2\alpha a - 3a)\tau + 6a^2 - 4\alpha a - 8a + 2]e^{-\tau} + \frac{a-1}{8a^3}[-(a^3-a^2)\eta^2 + 2(2\alpha a + 3a \\ &- 3)a\eta + 2(a^2 - 2\alpha a - 4a + 3)]e^{a\eta}\} + O(\epsilon^3), \end{aligned} \tag{3.31}$$

where $\tau = \frac{1-y}{\epsilon}$ and $\eta = \frac{1+y}{\epsilon}$.

3.2. Asymptotic solution for large suction Reynolds number R and large Hartmann number M

It is clear from (3.31) that the effect of magnetic field on channel flow can be negligible, hence, in this section, we will consider the case of large magnetic field, which may have noticeable effect to the flow. When the suction Reynolds number R and the Hartmann number M are both large and we take the same order of effect on the flow through the channel, there may exist a combined viscous suction and magnetic boundary layer at both channel walls. Hence, we assume $r = -\frac{M^2}{R} > 0$ and $r \sim O(1)$ and choose $\epsilon = -\frac{1}{R}$

as the perturbation parameter. The method used in Section 3.1 can be applied to this case. Equation (2.11) can be written as

$$\epsilon f''' + \epsilon\alpha(yf'' + 2f') - (ff'' - f'^2) + rf' = k, \tag{3.32}$$

where $\varepsilon K(R) = k$. A solution of (3.32) satisfying the corresponding boundary conditions (2.12) can be obtained by using the similar procedure as in the large suction case. Hence, the expressions of f_i, g_i and h_i are listed directly below in order to avoid tedious calculations and duplicatious:

$$f_0 = \frac{1-a}{2}y + \frac{1+a}{2}, \tag{3.33}$$

$$f_1 = -\frac{(a-1)^2}{4a}y + \frac{1-a^2}{4a}, \tag{3.34}$$

$$f_2 = \frac{a-1}{8a^3}[(3a^4+(2r-2\alpha-4)a^3+2a^2-(4+2\alpha)a-2r+3)y+3a^4+(2r-2\alpha-4)a^3+(4+2\alpha)a+2r-3], \tag{3.35}$$

$$g_1 = \frac{a-1}{2}e^{-\tau}, \tag{3.36}$$

$$h_1 = \frac{a-1}{2a}e^{a\eta}, \tag{3.37}$$

$$g_2 = \frac{a-1}{8a}[-(a^2-a)\tau^2 - 2(3a^2+(2r-2\alpha-3)a)\tau - 2(3a^2+2(r-\alpha-2)a+1)]e^{-\tau}, \tag{3.38}$$

$$h_2 = \frac{a-1}{8a^3}[-(a^3-a^2)\eta^2 + 2((3+2\alpha)a^2+(2r-3)a)\eta + 2(a^2-(4+2\alpha)a-2r+3)]e^{a\eta}. \tag{3.39}$$

Then the asymptotic solution for this case is given as follow:

$$\begin{aligned} f(y) =& \frac{1-a}{2}y + \frac{1+a}{2} + \epsilon[-\frac{(a-1)^2}{4a}y + \frac{1-a^2}{4a} + \frac{a-1}{2}e^{-\tau} + \frac{a-1}{2a}e^{a\eta}] + \epsilon^2\{\frac{a-1}{8a^3}[(3a^4 \\ &+ (2r-2\alpha-4)a^3 + 2a^2 - (4+2\alpha)a - 2r + 3)y + 3a^4 + (2r-2\alpha-4)a^3 + (4+2\alpha)a \\ &+ 2r - 3] + \frac{a-1}{8a}[-(a^2-a)\tau^2 - 2(3a^2+(2r-2\alpha-3)a)\tau - 2(3a^2+2(r-\alpha-2)a \\ &+ 1)]e^{-\tau} + \frac{a-1}{8a^3}[-(a^3-a^2)\eta^2 + 2((3+2\alpha)a^2+(2r-3)a)\eta + 2(a^2-(4+2\alpha)a-2r \\ &+ 3)]e^{a\eta}\} + O(\epsilon^3), \end{aligned} \tag{3.40}$$

where $\tau = \frac{1-y}{\epsilon}$, $\eta = \frac{1+y}{\epsilon}$ and $-1 < a < 0$. From (3.40), we know that the large magnetic field has noticeable effect on the flow.

4. Asymptotic solutions for mixed cases

For the asymmetric model, we will consider the case where the injection and suction are mixed at the upper and lower walls. The mixed cases are either mixed injection or mixed suction for the porous channel flow which was considered by Terril [16]. The flow governed by (2.11) is of mixed injection for positive values v_1 and v_2, while the flow is of mixed suction for negative values of v_1 and v_2. Both of the asymptotic expressions are presented in the subsequent subsections respectively.

4.1. Asymptotic solution for the mixed injection case

For the mixed injection case, assuming $v_2 \geq v_1 > 0$ (i.e. $0 < a \leq 1$), the equation, satisfying the condition (2.12), can be written as

$$\epsilon f''' + \epsilon\alpha(yf'' + 2f') + (ff'' - f'^2) - \epsilon M^2 f' = k, \tag{4.1}$$

where $\epsilon = \frac{1}{R} > 0$ can be treated as a small parameter and k is an arbitrary constant.

The wall dimensionless height $y = -1$ is with suction which may produce the boundary layer, while the other wall $y = 1$ is with injection. One correction term may have to be introduced due to the presence of boundary layer at $y = -1$. Thus, $f(y)$ and k are expanded as

$$f(y) = f_0(y) + \epsilon(f_1(y) + h_1(\eta)) + \epsilon^2(f_2(y) + h_2(\eta)) + \cdots, \tag{4.2}$$

$$k = k_0 + \epsilon k_1 + \epsilon k_2 + \cdots, \tag{4.3}$$

where $\eta = \frac{1+y}{\epsilon}$ is the stretching transformation near $y = -1$ and $h_i(\eta), i = 1, 2\cdots$ are boundary layer functions (rapidly decay when y is away from the wall $y = -1$). By substituting (4.2) into (2.12), the boundary conditions become

$$f_0|_{y=1} = 1,\ f_0'|_{y=1} = 0,\ f_0|_{y=-1} = 1 - \alpha_2 = a, \tag{4.4}$$

$$f_{i-1}'|_{y=-1} + \dot{h}_i|_{\eta=0} = 0,\ i = 1, 2, \cdots, \tag{4.5}$$

$$f_i|_{y=1} = 0,\ f_i'|_{y=1} = 0,\ f_i|_{y=-1} + h_i|_{\eta=0} = 0,\ i = 1, 2, \cdots, \tag{4.6}$$

Substituting (4.2) and (4.3) into (4.1) yields, at $O(1)$,

$$f_0 f_0'' - f_0'^2 = k_0 \tag{4.7}$$

subject to (4.4).

It can be easily verified that the leading order solution is $f_0 = \cos(by - b)$ which is a periodic function, then, we obtain $b = \frac{\cos^{-1} a + 2n\pi}{2}$ and $k_0 = -b^2 = -\frac{(\cos^{-1} a + 2n\pi)^2}{4}$, $n = 0, 1, 2\cdots$. Thus, we obtain many solutions of (4.7). f_0' which is proportional to the streamwise velocity u does not change sign for $n = 0$. However, f_0' does change sign at least three times for $n = 1$(i.e. the streamwise fluid flow direction will change at least three times). f_0' does change sign at least five times for $n \geq 2$, these phenomena may not occur physically. Laboratory experiment on porous pipe without expansion or contracting has been conducted by Wageman and Guevara [27] who obtain zero order asymptotic solution $f_0 = \frac{(-1)^n}{B} \sin\frac{(2n+1)\pi X}{2}$, $n = 0, 1, 2...$ for injection case. The only solution that has been experimentally observed was that case $n = 0$ in Guevara's work. Meanwhile, Proudman[14] pointed out that f can have at most one zero which is common to any combination of signs of v_1 and v_2 for the reduced solution. However, there is at least two zeros for $n > 0$. Therefore, there must be $n = 0$.

Now we consider $f_0 = \cos(by - b)$ as the leading order solution. When the terms of $O(\epsilon^{-1})$ are collected, the equation for h_1 becomes

$$\dddot{h}_1 + f_0(-1)\ddot{h}_1 = 0, \tag{4.8}$$

and satisfies the boundary condition (4.5)

$$\dot{h}_1|_{\eta=0} = -f_0'|_{y=-1} = -b\sin 2b. \tag{4.9}$$

Hence, the solution h_1 is

$$h_1 = \frac{b}{a}\sin 2b \cdot e^{-a\eta}. \tag{4.10}$$

When terms of $O(\epsilon)$ are collected, the differential equation for f_1 becomes

$$f_0 f_1'' - 2f_0' f_1' + f_0'' f_1 = -f_0''' - \alpha(y f_0'' + 2f_0') + M^2 f_0' + k_1. \tag{4.11}$$

The boundary conditions (4.6) are

$$f_1|_{y=1} = 0,\ \ f_1'|_{y=1} = 0, f_1|_{y=-1} = -h_1|_{\eta=0} = -\frac{b}{a}\sin 2b. \tag{4.12}$$

To simplify the equation, let $z = by - b$. Then, (4.11) can be written as

$$b\cos z f_1'' + 2b\sin z f_1' - b\cos z f_1 = (b+z)\alpha\cos z + (2\alpha - M^2 - b^2)\sin z + \lambda, \tag{4.13}$$

where ' denotes the derivative with respect to z and $\lambda = \frac{k_1}{b}$, and the boundary conditions become

$$f_1|_{z=0} = 0,\ \ f_1'|_{z=0} = 0,\ \ f_1|_{z=-2b} = -\frac{b}{a}\sin 2b. \tag{4.14}$$

To construct the solution of (4.13), start with the corresponding homogeneous equation

$$b\cos z f_{1h}'' + 2b\sin z f_{1h}' - b\cos z f_{1h} = 0. \tag{4.15}$$

We can easily obtain its solution

$$f_{1h} = P_1 \sin z + P_0(z\sin z + \cos z), \tag{4.16}$$

where P_0 and P_1 are two arbitrary constants. Applying the method of variation of parameters, we look for the solution of (4.16) in the terms of

$$f_1 = P_1(z)\sin z + P_0(z)(z\sin z + \cos z). \tag{4.17}$$

Through the standard process, it is obvious that

$$P_0 = \frac{1}{2b}[(b^2+M^2-4\alpha)(\ln(1-\sin z) - \ln\cos z) + \sec^2 z(b^2+M^2-2\alpha)\sin z - \lambda - 2\alpha(b+z)\cos z + (b^2)] + n_0, \tag{4.18}$$

$$P_1 = \frac{1}{2b}(-2(b^2+M^2-2\alpha-b\alpha z)\sec z + \lambda z\sec^2 z + \lambda\tan z) + Q(z) + n_1, \tag{4.19}$$

where

$$Q(z) = \frac{\alpha}{b}z^2\sec z + \frac{1}{b}\int_0^z (b^2+M^2+\alpha)\phi\sec\phi - (b^2+M^2-2\alpha)\phi\sec^3\phi \mathrm{d}\phi, \tag{4.20}$$

and n_0 and n_1 are constants to be determined using the boundary conditions. Substituting (4.18) and (4.19) into (4.17) yields

$$\begin{aligned} f_1 =& -\alpha - \frac{\alpha}{b}z + (Q(z)+n_1)\sin z + n_0 z\sin z + n_0\cos z - \frac{b^2+M^2-2\alpha}{2b}\tan z - \frac{\alpha}{b}\tan z \\ &+ \frac{\lambda}{2b}\sin z\tan z - \frac{\lambda}{2b}\sec z + \frac{b^2+M^2-2\alpha}{2b}z\tan^2 z + \frac{b^2+M^2-4\alpha}{2b}(\ln(1-\sin z) \\ &- \ln\cos z)(z\sin z + \cos z). \end{aligned} \tag{4.21}$$

According to the boundary conditions (4.14) and taking account of $Q(0) = 0$, we have

$$n_0 = \alpha + \frac{\lambda}{2b},\quad n_1 = \frac{b^2+M^2-2\alpha}{b}, \tag{4.22}$$

$$\begin{aligned}\lambda =&\frac{1}{2a(b\sin 2b+\cos 2b)}(2ab\alpha+(2b^2-2ab^2-2aM^2+4a\alpha-4ab^2\alpha-2abQ(-2b))\sin 2b\\&-2ab\alpha\cos 2b+(ab^2+aM^2-2a\alpha+8ab^2\alpha)\tan 2b+(-2ab^3-2abM^2+4ab\alpha)\tan^2 2b\\&+a(b^2+M^2-4\alpha)(\cos 2b+2b\sin 2b)(\ln(1+\sin 2b)-\ln\cos 2b)).\end{aligned}\tag{4.23}$$

Hence, the solution of equation (4.11) is

$$\begin{aligned}f_1 =&-\alpha-\frac{\alpha}{b}z+(Q(z)+\frac{b^2+M^2-2\alpha}{b})\sin z+(\alpha+\frac{\lambda}{2b})z\sin z+(\alpha+\frac{\lambda}{2b})\cos z\\&-\frac{b^2+M^2-2\alpha+2\alpha z^2}{2b}\tan z+\frac{\lambda}{2b}\sin z\tan z-\frac{\lambda}{2b}\sec z+\frac{b^2+M^2-2\alpha}{2b}z\tan^2 z\\&+\frac{b^2+M^2-4\alpha}{2b}(\ln(1-\sin z)-\ln\cos z)(z\sin z+\cos z),\end{aligned}\tag{4.24}$$

where λ is given by (4.23). Finally, an asymptotic solution for the mixed injection for $v_2\geq v_1>0$ (i.e. $0<a\leq 1$) is

$$\begin{aligned}f(y) =&\cos z+\epsilon\{-\alpha-\frac{\alpha}{b}z+(Q(z)+\frac{b^2+M^2-2\alpha}{b})\sin z+(\alpha+\frac{\lambda}{2b})z\sin z+(\alpha+\frac{\lambda}{2b})\cos z\\&-\frac{b^2+M^2-2\alpha+2\alpha z^2}{2b}\tan z+\frac{\lambda}{2b}\sin z\tan z-\frac{\lambda}{2b}\sec z+\frac{b^2+M^2-2\alpha}{2b}z\tan^2 z\\&+\frac{b^2+M^2-4\alpha}{2b}(\ln(1-\sin z)-\ln\cos z)(z\sin z+\cos z)+\frac{b}{a}\sin 2b\cdot e^{-a\eta}\}+O(\epsilon^2),\end{aligned}\tag{4.25}$$

where $\eta=\frac{1+y}{\epsilon}$ and $z=by-b$.

Remark: We remark that $v_1\geq v_2>0$ (i.e. $a\geq 1$), we can have the solution of format $f_0=\cosh(dy-d)$, where $d=\pm\frac{\cosh^{-1}a}{2}$ and $k_0=d^2=\frac{(\cosh^{-1}a)^2}{4}$. Since $\cosh(x)$ is an even function, the solution for $d=\frac{\cosh^{-1}a}{2}$ and $d=-\frac{\cosh^{-1}a}{2}$ are the same, and it is also means that there is only one solution of (4.7).

4.2. Asymptotic solution for mixed suction case

For the mixed suction case, assuming $v_2\leq v_1<0$ (i.e. $0<a\leq 1$), the equation is the same as (3.1), satisfying the condition (2.12). Correction terms may have to be introduced due to a possible boundary layer near $y=1$. Thus, $f(y)$ and k are expanded as

$$f(y) = f_0(y)+\epsilon(f_1(y)+g_1(\tau))+\epsilon^2(f_2(y)+g_2(\tau))+\cdots,\tag{4.26}$$
$$k = k_0+\epsilon k_1+\epsilon k_2+\cdots,\tag{4.27}$$

where $\epsilon=-\frac{1}{R}>0$, $\tau=\frac{1-y}{\epsilon}$ is the stretching transformation near $y=1$ and $g_i(\tau), i=1,2\cdots$ are boundary layer functions (rapidly decay when y is away from $y=1$). The corresponding boundary conditions become

$$f_0|_{y=1}=1, f_0|_{y=-1}=1-\alpha_2=a, f_0'|_{y=-1}=0,\tag{4.28}$$
$$f_{i-1}'|_{y=1}-\dot{g}_i|_{\tau=0}=0, i=1,2,\cdots,\tag{4.29}$$
$$f_i|_{y=-1}=0, f_i'|_{y=-1}=0, f_i|_{y=1}+g_i|_{\tau=0}=0, i=1,2,\cdots,\tag{4.30}$$

Substituting (4.26) and (4.27) into (3.1) and collecting the same power of ϵ yield

$$\begin{aligned}
\epsilon^0 &: f_0'^2 - f_0 f_0'' = k_0, && (4.31)\\
\epsilon^{-1} &: \dddot{g}_1 + \ddot{g}_1 = 0, && (4.32)\\
\epsilon^1 &: f_0 f_1'' - 2f_0' f_1' + f_0'' f_1 = \alpha(y f_0'' - 2f_0') - M^2 f_0' + f_0''' - k_1, && (4.33)\\
&\cdots . && (4.34)
\end{aligned}$$

The above equations (4.31)-(4.33) subject to boundary conditions (4.28)-(4.30), respectively. It can be easily obtained that the leading order solution is

$$f_0 = a\cosh(by + b), \tag{4.35}$$

where $b = \frac{1}{2}\cosh^{-1}\frac{1}{a}$ and $k_0 = -(ab)^2 = -\frac{1}{4}a^2(\cosh^{-1}\frac{1}{a})^2$.
Applying the similar method in the mixed injection case and solving (4.32) and (4.33), one obtains

$$g_1 = -ab\sinh 2b \cdot e^{-\tau}, \tag{4.36}$$

$$\begin{aligned}
f_1(z) =& -\alpha + \frac{\alpha}{b}z - (S(z) + \frac{M^2 - b^2 - 2\alpha}{b})\sinh z - \frac{\mu}{2b}\cosh z - \frac{b^2 - M^2 + 2\alpha + 2\alpha z^2}{2b}\tanh z\\
& - \frac{b^2 - M^2 + 2\alpha}{2b} z\tanh^2 z - \frac{2b\alpha + \mu - 2(b^2 - M^2 + 4\alpha)\tan^{-1}\tanh z/2}{2b}(z\sinh z - \cosh z),
\end{aligned} \tag{4.37}$$

where

$$S(z) = \frac{1}{b}\int_0^z -\alpha\phi\mathrm{sech}\phi + \alpha\phi^2\mathrm{sech}\phi\tanh\phi + (b^2 - M^2 + 2\alpha)\phi\mathrm{sech}\phi\tanh^2\phi\, d\phi, \tag{4.38}$$

and

$$\begin{aligned}
\mu =& b(1 - a - 2\alpha) - S(2b) - \frac{M^2 - 2\alpha}{b} + \frac{M^2 - b^2 - 2\alpha - 8b^2\alpha}{2b\cosh 2b} + \alpha\coth 2b + (M^2 - b^2 - 2\alpha)\\
&\cdot \frac{\tanh 2b}{\cosh 2b} + \frac{\alpha}{2\sinh b\cosh 2b} + \frac{M^2 - b^2 - 4\alpha}{b}\arctan\tanh b(\coth 2b - 2b).
\end{aligned} \tag{4.39}$$

Finally, an asymptotic solution of mixed suction for $v_2 \le v_1 < 0$ (i.e. $0 < a \le 1$) is

$$\begin{aligned}
f(y) =& \cosh z + \epsilon\{-\alpha + \frac{\alpha}{b}z - (S(z) + \frac{M^2 - b^2 - 2\alpha}{b})\sinh z - \frac{\mu}{2b}\cosh z - \frac{b^2 - M^2 + 2\alpha + 2\alpha z^2}{2b}\\
&\cdot\tanh z - \frac{b^2 - M^2 + 2\alpha}{2b} z\tanh^2 z - \frac{2b\alpha + \mu - 2(b^2 - M^2 + 4\alpha)\tan^{-1}\tanh z/2}{2b}(z\sinh z\\
&- \cosh z) - ab\sinh 2b \cdot e^{-\tau}\} + O(\epsilon^2),
\end{aligned} \tag{4.40}$$

where $\tau = \frac{1-y}{\epsilon}$ and $z = by + b$.

Remark: we remark that $v_1 \le v_2 < 0$ (i.e. $a \ge 1$), $a \ge 1$, $f_0 = a\cos(dy + d)$ is a leading order solution, where $b = \frac{1}{2}(\cos^{-1}\frac{1}{a} + 2n\pi)$ and $k_0 = (ab)^2 = \frac{1}{4}a^2(\cos^{-1}\frac{1}{a} + 2n\pi)^2$.

5. Asymptotic solution for the case of small Reynolds R and small expansion ratio α case

For the small injection or suction and small expansion ratio case, it is reasonable to use the method of regular perturbation expansions to construct the asymptotic solution. One, thus, may treat R as a perturbation parameter. The solution may be expanded as

$$f = f_0 + Rf_1 + R^2 f_2 + \cdots . \tag{5.1}$$

Differentiating (2.11) with respect to y, one obtains

$$f^{iv} + \alpha(yf''' + 3f'') + R(f'''f - f''f') - M^2 f'' = 0 \tag{5.2}$$

satisfying the boundary condition (2.12). Substituting (5.1) into (5.2) and equating the like powers of R yield

$$f_0^{iv} + \alpha(yf_0''' + 3f_0'') - M^2 f_0'' = 0, \tag{5.3}$$

$$f_1^{iv} + \alpha(yf_1''' + 3f_1'') - M^2 f_1'' + f_0'''f_0 - f_0''f_0' = 0. \tag{5.4}$$

The corresponding boundary conditions become

$$f_0|_{y=1} = 1, \qquad f_0'|_{y=1} = 0, \qquad f_0|_{y=-1} = 1 - \alpha_2, \qquad f_0'|_{y=-1} = 0, \tag{5.5}$$

$$f_1|_{y=1} = 0, \qquad f_1'|_{y=1} = 0, \qquad f_1|_{y=-1} = 0, \qquad f_1'|_{y=-1} = 0. \tag{5.6}$$

We consider the case where α is also small and take it as the secondary perturbation parameter. Then, f_0 and f_1 can be further expanded in the form

$$f_0 = f_{00} + \alpha f_{01} + O(\alpha^2), \tag{5.7}$$

$$f_1 = f_{10} + \alpha f_{11} + O(\alpha^2). \tag{5.8}$$

Substituting (5.7) into (5.3) and collecting like powers of α yield

$$f_{00}^{(iv)} - M^2 f_{00}'' = 0, \tag{5.9}$$

$$f_{01}^{(iv)} - M^2 f_{01}'' + yf_{00}''' + 3f_{00}'' = 0, \tag{5.10}$$

with the boundary conditions

$$f_{00}|_{y=1} = 1, \qquad f_{00}'|_{y=1} = 0, \qquad f_{00}|_{y=-1} = 1 - \alpha_2, \qquad f_{00}'|_{y=-1} = 0, \tag{5.11}$$

$$f_{01}|_{y=1} = 0, \qquad f_{01}'|_{y=1} = 0, \qquad f_{01}|_{y=-1} = 0, \qquad f_{01}'|_{y=-1} = 0. \tag{5.12}$$

Thus, the results are

$$f_{00} = \frac{\alpha_2(\sinh My - yM\cosh M)}{2(\sinh M - M\cosh M)} + \frac{2-\alpha_2}{2}, \tag{5.13}$$

$$\begin{aligned} f_{01} =& D/(8M)[y^2(M\cosh M - \sinh M)M\sinh My - y((M\cosh M - \sinh M)\cosh My \\ &+ \cosh M\sinh M + M\cosh 2M - 2M) - (M\cosh M + 2\sinh M)M\sinh My], \end{aligned} \tag{5.14}$$

where $D = \alpha_2(M\cosh M - \sinh M)^{-2}$.
Substituting (5.8) and (5.7) into (5.4) and equating equal powers of α, the leading order equation becomes

$$f_{10}^{(iv)} - M^2 f_{10}'' + f_{00}'''f_{00} - f_{00}''f_{00}' = 0 \tag{5.15}$$

with the boundary condition

$$f_{10}|_{y=1} = 0, f_{10}'|_{y=1} = 0, f_{10}|_{y=-1} = 0, f_{10}'|_{y=-1} = 0. \tag{5.16}$$

The solution of f_{10} is

$$
\begin{aligned}
f_{10} =& Gy^2[2\alpha_2 M^2 \sinh 2M \sinh My(M\cosh M - \sinh M)] + MGy[8(\alpha_2 - 2)(M\cosh M - \sinh M)^2 \\
&\cdot \sinh My + 28\alpha_2 \cosh M(M\cosh M - \sinh M)\cosh My + \alpha_2(14\sinh 2M\cosh M - 2M \\
&\cdot \cosh 3M - 26M\cosh M)] + 2G\{4(\alpha_2 - 2)\sinh^2 M[M\cosh^2 M - \cosh M(M\cosh My \\
&+ \sinh M) + \sinh M(\cosh My + M\sinh M)] - M^2 \sinh 2M[(\alpha_2 - 2)(2\cosh M\cosh My \\
&+ \cosh 2M) - 3\alpha_2(2\sinh M\sinh My + 1) + 6] + 2M^3\cosh^2 M[2(\alpha_2 - 2)(\cosh M\cosh My \\
&- 1) - \alpha_2 \sinh M\sinh My]\},
\end{aligned} \tag{5.17}
$$

where $G = \alpha_2(64M\sinh M)^{-1}(M\cosh M - \sinh M)^{-3}$. Finally, the solution becomes

$$
\begin{aligned}
f(y) =& \alpha_2(\sinh My - yM\cosh M)/(2p) + (2-\alpha_2)/2 + \alpha\alpha_2/(8Mp^2)[y^2 p - \sinh M)M\sinh My \\
&- y(p\cosh My + \cosh M\sinh M + M\cosh 2M - 2M) - (M\cosh M + 2\sinh M)M \\
&\cdot \sinh My] + R\alpha_2/(64p^3 M\sinh M)\{y^2[2\alpha_2 M^2\sinh 2M\sinh Myp] + My[8(\alpha_2-2)p^2 \\
&\cdot \sinh My + 28\alpha_2\cosh Mp\cosh My + \alpha_2(14\sinh 2M\cosh M - 2M\cosh 3M - 26M \\
&\cdot \cosh M)] + 2\{4(\alpha_2-2)\sinh^2 M[M\cosh^2 M - \cosh M(M\cosh My + \sinh M) \\
&+ \sinh M(\cosh My + M\sinh M)] - M^2\sinh 2M[(\alpha_2-2)(2\cosh M\cosh My \\
&+ \cosh 2M) - 3\alpha_2(2\sinh M\sinh My + 1) + 6] + 2M^3\cosh^2 M[2(\alpha_2-2)(\cosh M \\
&\cdot \cosh My - 1) - \alpha_2\sinh M\sinh My]\}\},
\end{aligned} \tag{5.18}
$$

where $p = M\cosh M - \sinh M$.

6. Comparison of the asymptotic and numerical solutions

Table 6.1: Comparison of $f''(-1)$ and $f''(1)$ for large suction R

M	α_2	α	R	$f''(-1)$ Asymptotic results	$f''(-1)$ Numerical results	$f''(1)$ Asymptotic results	$f''(1)$ Numerical results
1.0	1.87	-1.5	-50	40.9395	40.8663	-47.4089	-47.4060
1.0	1.95	-2.0	-67	62.9581	62.9526	-66.3744	-66.3825
1.1	1.73	-3.0	-88	56.8609	56.8746	-78.2435	-78.2997
3.5	1.81	2.0	-90	62.9612	62.9067	-79.0131	-79.0398
6.5	1.50	3.5	-99	32.8125	32.9871	-71.6250	-71.9587
3.8	1.77	4.0	-110	70.1684	70.0583	-93.2607	-93.2641
2.5	1.60	-8.0	-135	69.7067	69.9336	-114.1870	-114.3161
6.0	1.50	3.0	-150	52.3125	52.3826	-110.2500	-110.4426
20.0	1.70	5.0	-150	83.6582	86.8943	122.8371	-125.1102
30.0	1.80	-7.0	-150	113.0850	119.4959	-140.6930	-145.8802
20.0	1.70	5.0	-175	98.5332	101.2919	-144.0871	-146.0277
30.0	1.80	-7.0	-175	131.0850	136.6659	-163.1930	-167.6935

Terrill[15] pointed out that the comparison of $f''(1)$ was found to be the most effective way. Hence, in this section we will make comparison of the asymptotic and numerical results of $f''(-1)$ and $f''(1)$ which are proportional to the skin-friction at the walls. The numerical solutions for (2.11) and (2.12) can be readily obtained by a matlab boundary value problem solver bvp4c.

For the large suction case, it can be seen from Table 6.1, the analytical results agree well with the numerical results for arbitrary constants α_2 and α for $M^2/R \sim O(\epsilon)$. Nevertheless, it is clear from the last four lines of the Table 6.1 that asymptotic results deteriorate when M^2 becomes of the same order as that of the suction Reynolds number. Hence, we need to use the asymptotic solution for the case of $M^2/R \sim O(1)$. We find that the large magnetic field has noticeable effect on the solution, because the parameter r has went into the terms of $O(\epsilon)^2$. As shown in Table 6.2, when using the solution (3.40), the accuracy is largely improved compared with Table 6.1 for large M.

Table 6.2: Comparison of $f''(-1)$ and $f''(1)$ for large suction R and M

				$f''(-1)$		$f''(1)$	
M	α_2	α	R	Asymptotic results	Numerical results	Asymptotic results	Numerical results
20	1.7	5	-150	86.8963	86.8943	-125.1038	-125.1102
20	1.7	5	-175	101.3087	101.2919	-146.0300	-146.0277
30	1.8	-7	-150	119.835	119.4959	-146.0925	-145.8802
30	1.8	-7	-175	136.8707	136.6659	-167.8211	-167.6935
20	1.7	5	-213	123.4236	123.3904	-177.9834	-177.9750
30	1.8	-7	-213	188.9365	188.8691	-208.9649	-208.9114

For the mixed cases, as shown in Table 6.3 and Table 6.4, the asymptotic solutions realize a good agreement with the numerical solutions. Meanwhile, tabulated values indicate that the skin-friction is increasing with the increasing R.

Table 6.3: Comparison of $f''(-1)$ and $f''(1)$ for mixed injection

				$f''(-1)$		$f''(1)$	
M	α_2	α	R	Asymptotic results	Numerical results	Asymptotic results	Numerical results
2	0.2	-3	50	7.66931	7.8352	-0.1244	-0.1256
2	0.2	-3	70	10.7495	10.9157	-0.1184	-0.119
2	0.2	-3	90	13.8335	14.0000	-0.1151	-0.1155
2	0.2	-3	110	16.9193	17.0860	-0.1129	-0.1132
2	0.2	-3	130	20.0060	20.1729	-0.1115	-0.1117
2	0.2	-3	150	23.0933	23.2603	-0.1104	-0.1106
3	0.1	2	50	4.3808	4.2281	-0.0541	-0.0541
3	0.1	2	70	6.1493	5.9912	-0.0531	-0.0531
3	0.1	2	90	7.9181	7.7570	-0.0526	-0.0526
3	0.1	2	110	9.6872	9.5242	-0.0523	-0.0523
3	0.1	2	130	11.4564	11.2920	-0.0521	-0.052
3	0.1	2	150	13.2256	13.0602	-0.0519	-0.0519

At this juncture, it is useful to note that if there is no magnetic field (i.e.$M = 0$) and the physic model is symmetric (i.e.$\alpha_2 = 2$), the solution of small R and small α can be reduced to the solution given by Majdalani and Zhou[9]. Furthermore, it is clear from Table.6.5 that the asymptotic results have a good match to the numerical results and the accuracy is increasing with the decreasing of R and α.

Table 6.4: Comparison of $f''(-1)$ and $f''(1)$ for mixed suction

M	α_2	α	R	$f''(-1)$ Asymptotic results	$f''(-1)$ Numerical results	$f''(1)$ Asymptotic results	$f''(1)$ Numerical results
1	0.06	-3.7	-15	0.0485	0.0527	-0.9107	-0.9342
2	0.06	-3.7	-15	0.0527	0.0583	-0.9194	-0.9018
1	0.13	1.2	-19	0.0599	0.0606	-2.4275	-2.5332
2	0.13	1.2	-28	0.0662	0.0665	-3.6532	-3.6001
2	0.24	2.8	-39	0.1056	0.1065	-9.6410	-9.5938
2	0.20	-1.5	-50	0.0949	0.0955	-8.7021	-8.5348
1	0.36	0.8	-55	0.1632	0.1634	-21.1974	-21.3687
2	0.54	1.7	-67	0.2255	0.2259	-41.4631	-41.1129
2	0.08	-2.1	-75	0.0439	0.0440	-6.0502	-5.9984
1	0.49	0.9	-89	0.2095	0.2097	-49.0864	-49.3811

Table 6.5: Comparisons of $f''(-1)$ and $f''(1)$ for small R and small α

M	α_2	α	R	$f''(-1)$ Asymptotic results	$f''(-1)$ Numerical results	$f''(1)$ Asymptotic results	$f''(1)$ Numerical results
3	1.9	0.01	0.01	4.2349	4.2349	-4.2342	-4.2342
3	1.8	0.02	0.03	4.0035	4.0035	-3.9996	-3.9996
5	1.5	0.04	0.06	4.6636	4.6636	-4.6495	-4.6497
7	1.3	0.10	0.09	5.2665	5.2667	-5.2426	-5.2430
9	1.0	0.15	0.19	5.0262	5.0263	-4.9727	-4.9735
12	0.8	0.22	0.28	5.2042	5.2043	-5.1309	-5.1319
15	0.6	-0.35	-0.30	4.8575	4.8576	-4.9250	-4.9260
20	0.4	-0.40	0.29	4.2726	4.2730	-4.2238	-4.2238
13	0.3	0.50	-0.38	2.0482	2.0490	-2.1007	-2.1007
8	0.1	-0.30	0.40	0.4765	0.4768	-0.4548	-0.4548
8	0.1	0.50	0.60	0.4580	0.4581	-0.4255	-0.4254

7. Conclusion

In this paper, we construct the similarity solutions of asymmetric channel flow with porous retractable walls and a transverse magnetic field. First, one asymptotic solution with the linear leading order is obtained for the large suction case, and it is found that the skin-friction near the walls increases with the increasing magnetic field intensity. Secondly, we obtain asymptotic solutions for the flow in a channel that one wall is with injection and the other is with suction. All above asymptotic solutions are constructed for the most difficult large Reynolds number cases. Finally, when the wall contraction or expansion is weak and the injection or suction is small, one asymptotic solution is obtained by the two-parameter perturbation method. All asymptotic solutions are verified by numerical solutions obtained by a Matlab boundary value problem solver bvp4c.

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